Electrical and Electronics Engineering
2024-2025
Master Semester 2

Course
Smart grids technologies

DFT-based Synchrophasor Estimation Algorithms - Spectral Leakage

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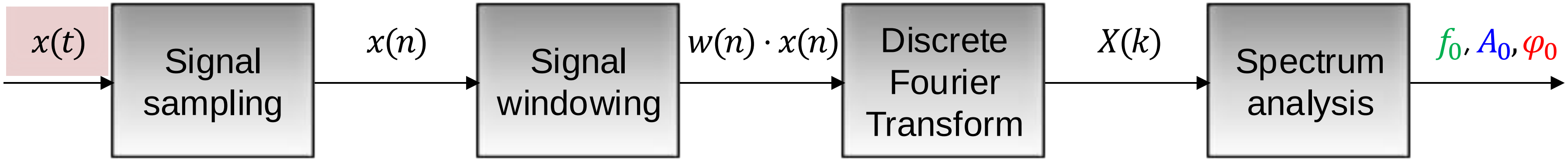
Outline

- The DFT measurement chain
  - Signal sampling
  - Signal windowing
  - DFT
- Spectral leakage
  - Analytical derivation
- Special windowing functions

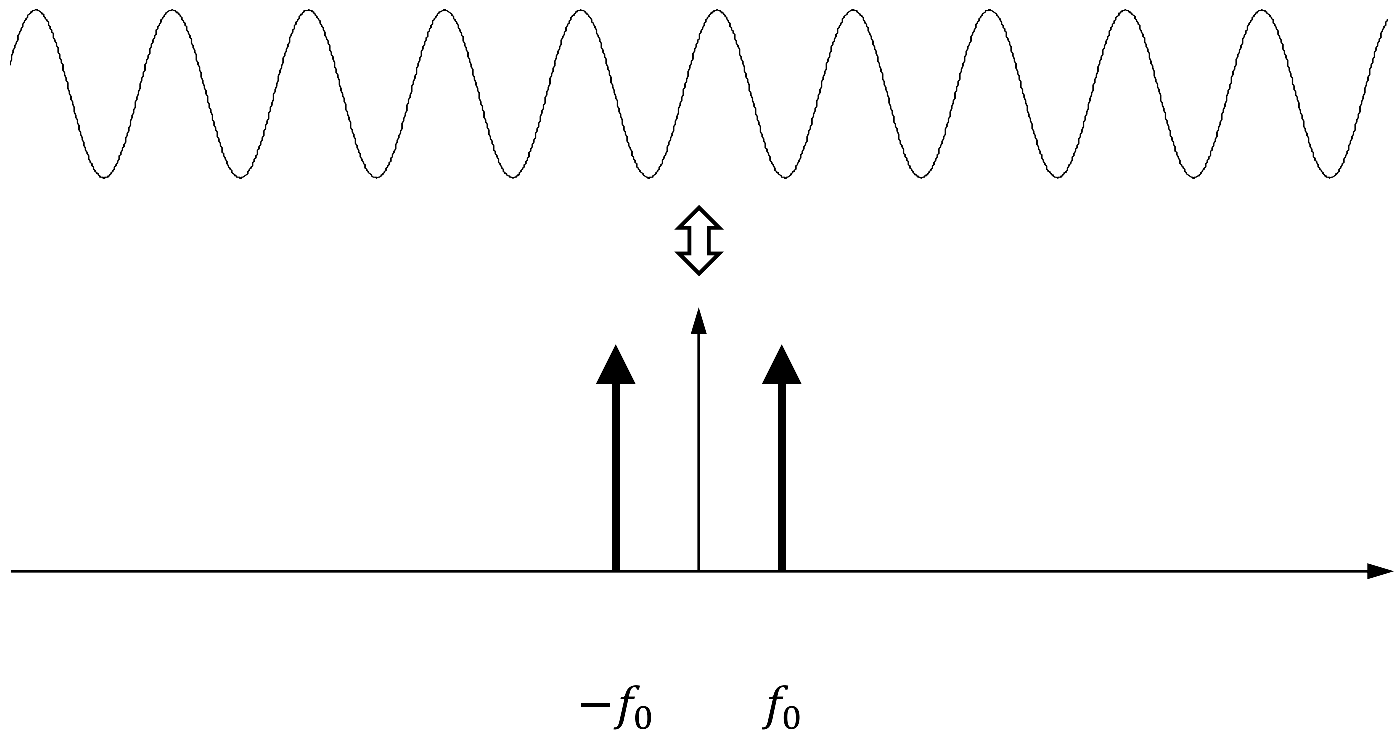
The DFT measurement chain

Continuous signal



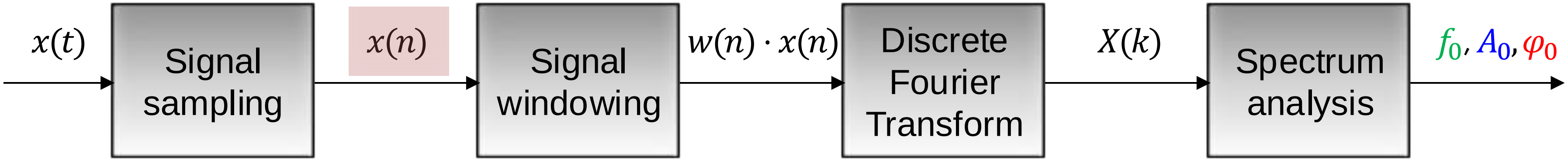
Signal:

Fourier Transform:



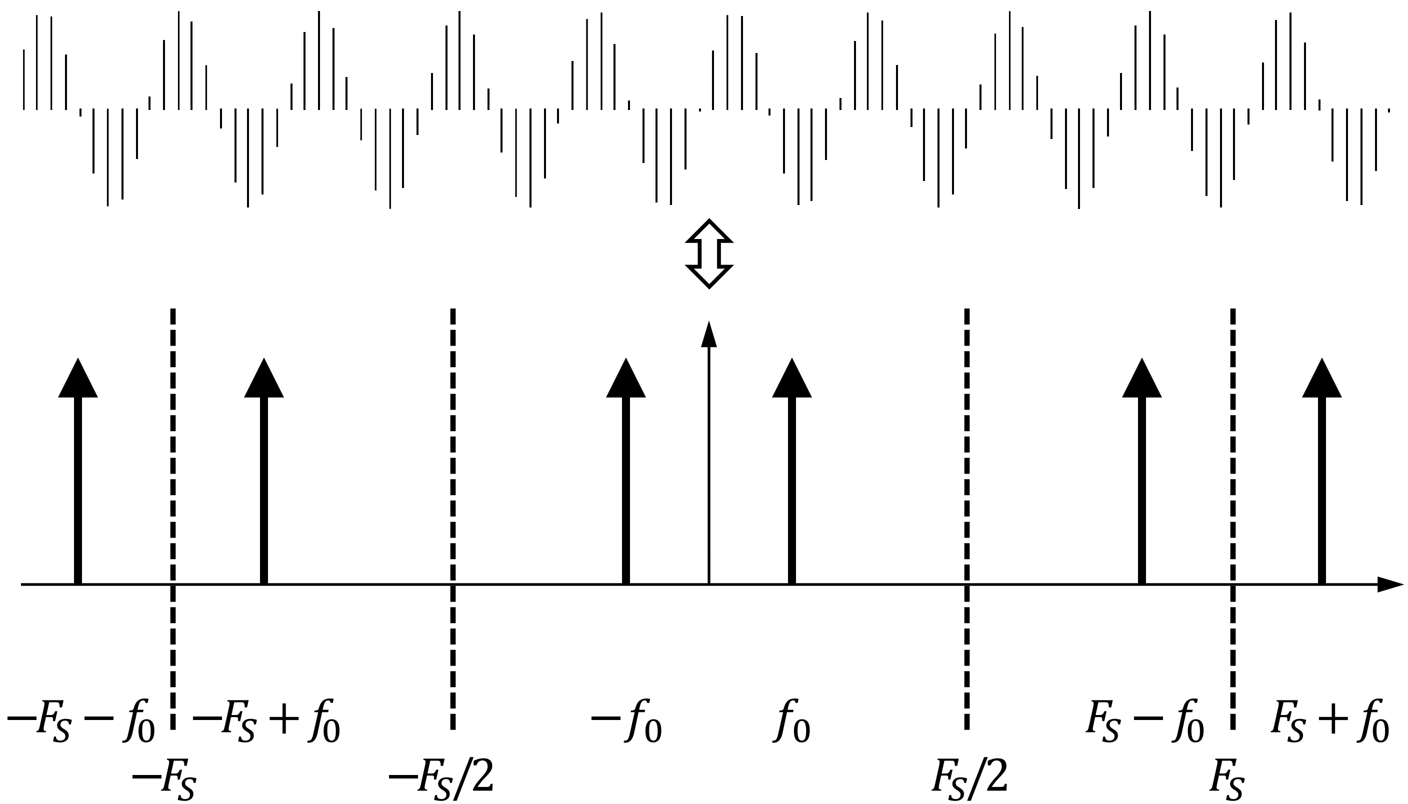
The DFT measurement chain

Sampled signal



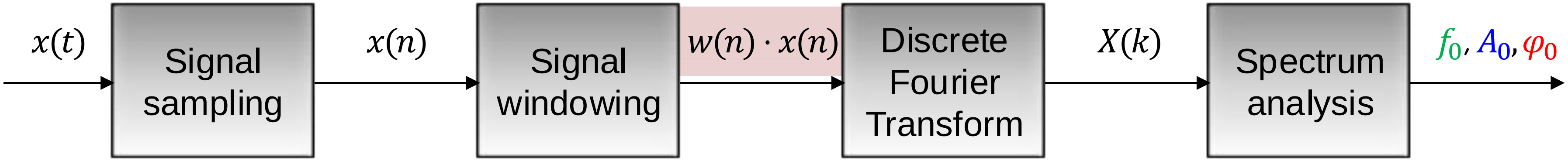
Signal:

Fourier Transform:

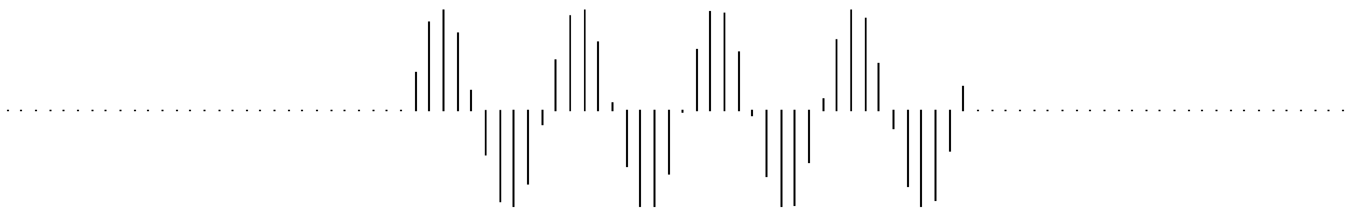


The DFT measurement chain

Windowed signal



Signal:



⇳

Fourier Transform:

?

Signal windowing

The Fourier transform of the rectangular window

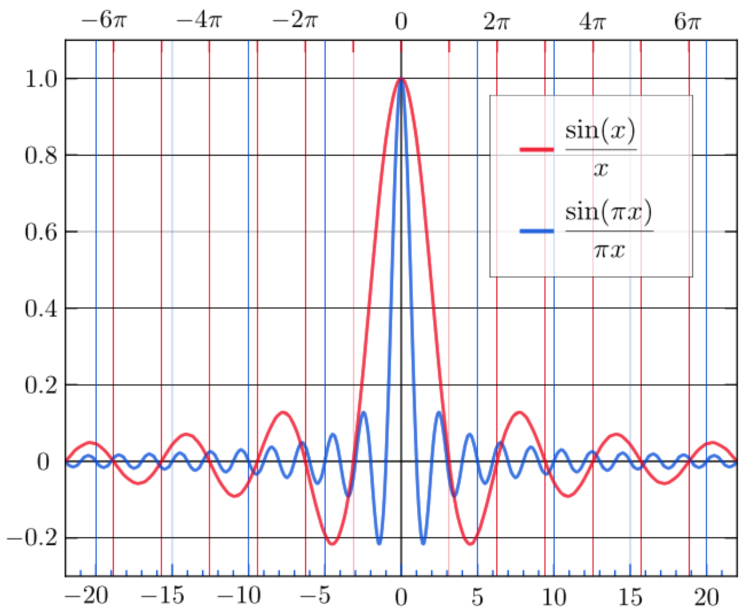
$$\Im\{w_R(t)\} = W_R(f) = \int_{-\infty}^{\infty} w(t) \cdot e^{-j2\pi f t} dt$$

$$= \int_{-T/2}^{+T/2} 1 \cdot e^{-j2\pi f t} dt = \frac{1}{-j2\pi f}\left[e^{-j2\pi f t}\right]_{-T/2}^{+T/2}$$

$$= \frac{1}{-j2\pi f}\left[e^{-j2\pi f T/2} - e^{j2\pi f T/2}\right]$$

$$= \frac{1}{\pi f}\left[\frac{e^{j\pi f T} - e^{-j\pi f T}}{2j}\right] = \frac{1}{\pi f}\sin(\pi f T)$$

$$= T\frac{\sin(\pi f T)}{\pi f T} \triangleq T\,\mathrm{sinc}(fT)$$



The sinc function is characterized by a vey important property:

$$\mathrm{sinc}(fT)\Big|_{f=1/T} = 0$$

MEANING: the zero-crossing of the Fourier transform of the rectangular window are equally spaced and happens at multiple of $1/T$.

Signal windowing

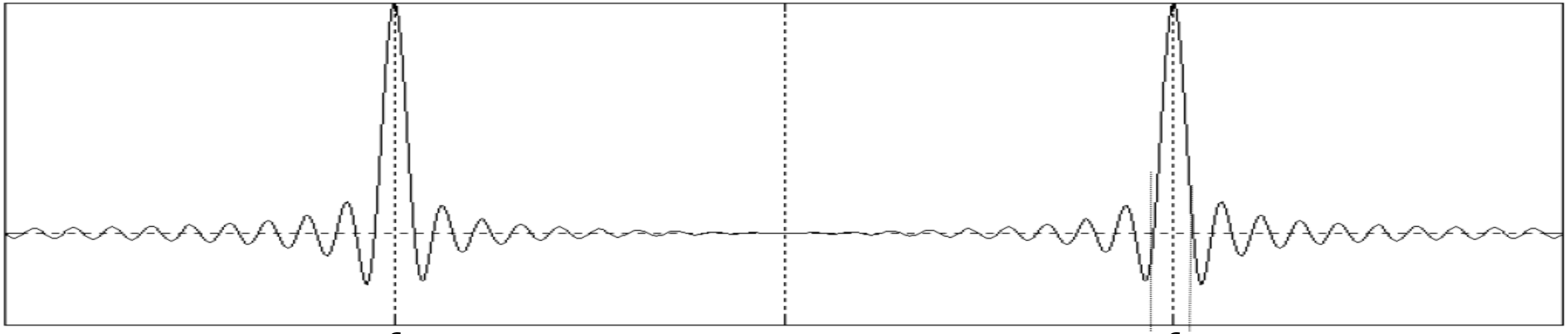
The Fourier transform of the windowed signal

Based on the convolution theorem, the Fourier transform of the windowed signal can be obtained as the **convolution** between the Fourier transforms of the rectangular window and the Fourier transform of the acquired signal (for the sake of simplicity let's model the signal as an ideal sinusoidal waveform):

$$\Im\{w_R(t)\cdot x(t)\} = \Im\{w_R(t)\cdot \cos(2\pi f_0 t)\} = W_R(f) * X(f)$$

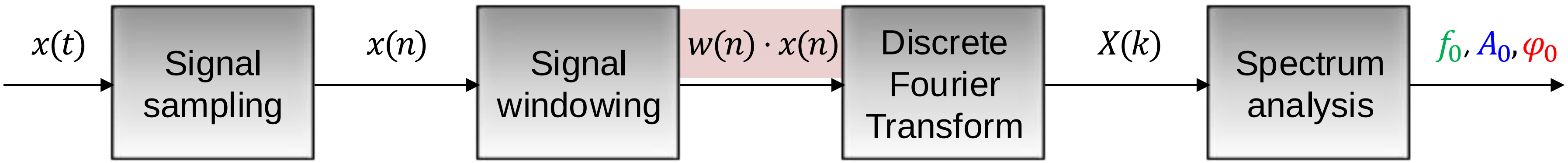
$$= T\,\mathrm{sinc}(fT) * \frac{1}{2}[\delta(f-f_0)+\delta(f+f_0)] = \frac{T}{2}\left[\mathrm{sinc}\big((f-f_0)T\big)+\mathrm{sinc}\big((f+f_0)T\big)\right]$$

that means that the convolution between the Fourier transform of the window function and that of the signal, produces two sinc (also called *images*) **translated around** $\pm f_0$.



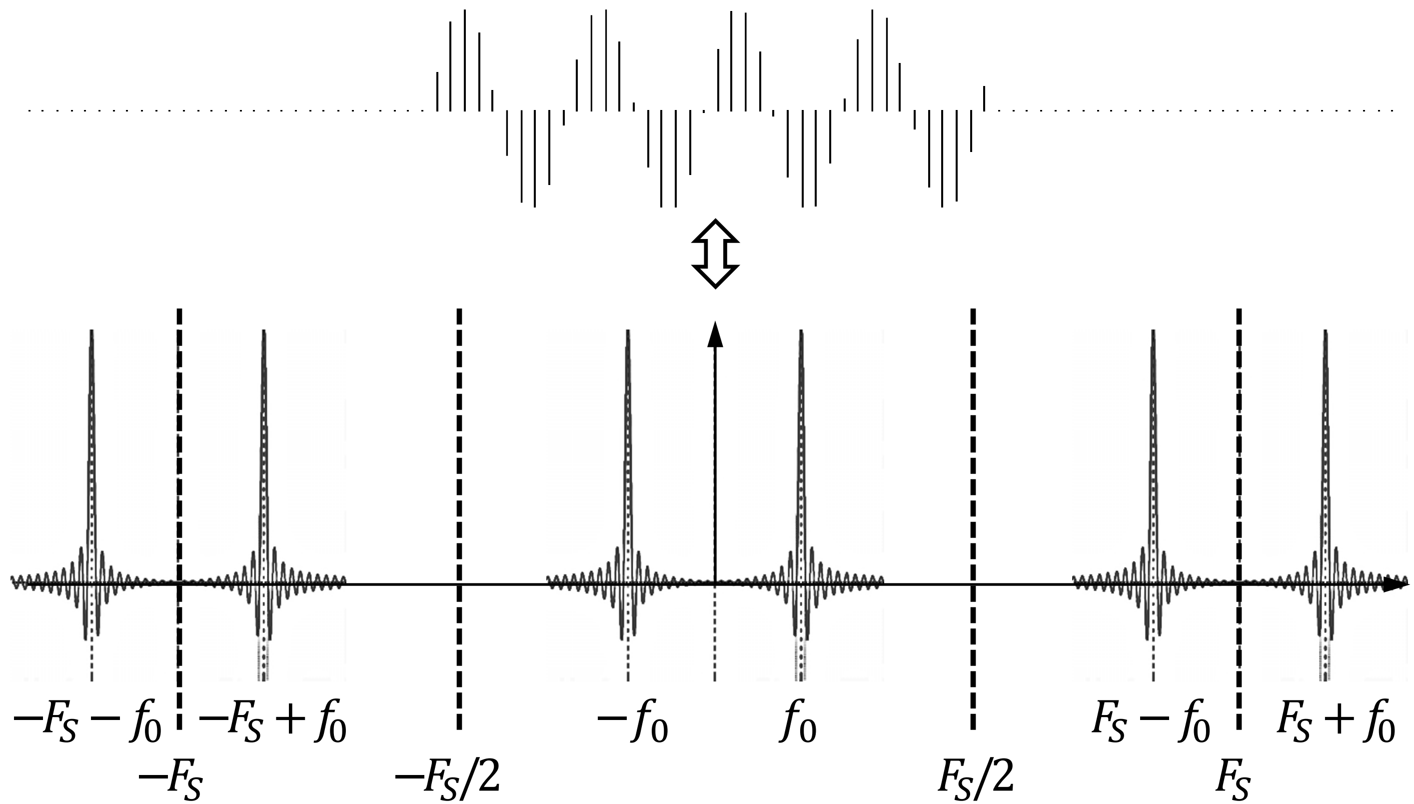
The DFT measurement chain

Windowed signal



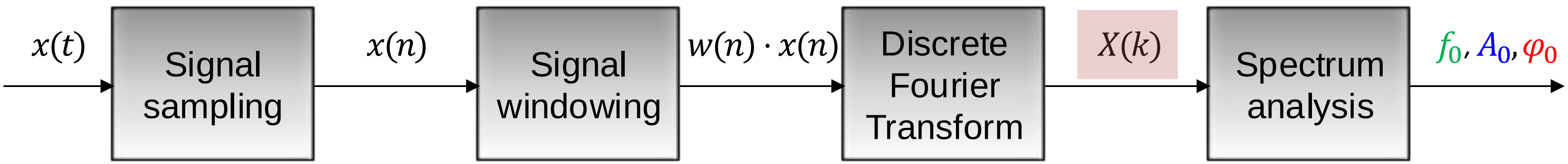
Signal:

Fourier Transform:

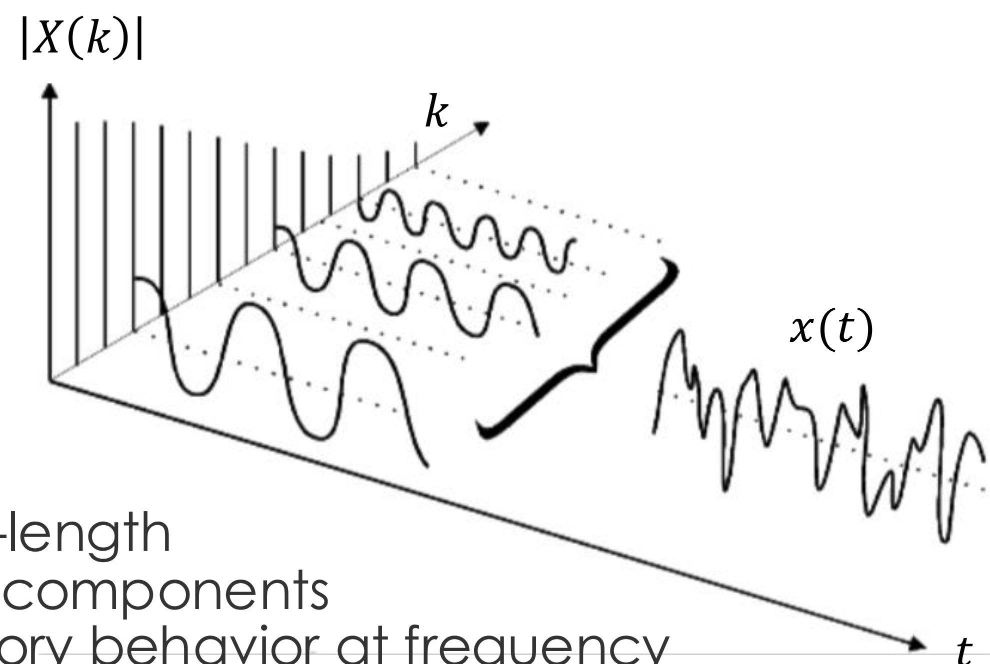


The DFT measurement chain

Discrete Fourier Transform



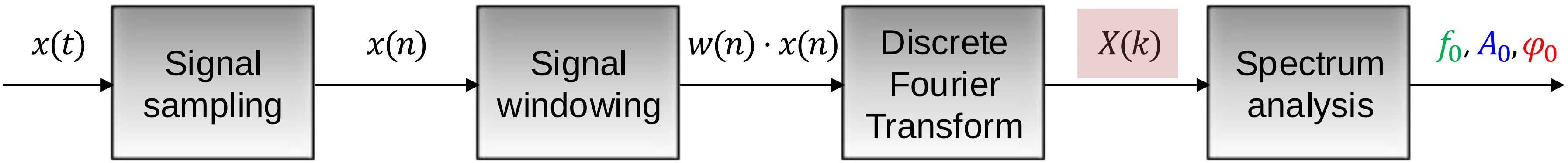
$$X(k) \triangleq \frac{2}{B} \sum_{n=0}^{N-1} w(n)x(n)W_N^{kn}$$



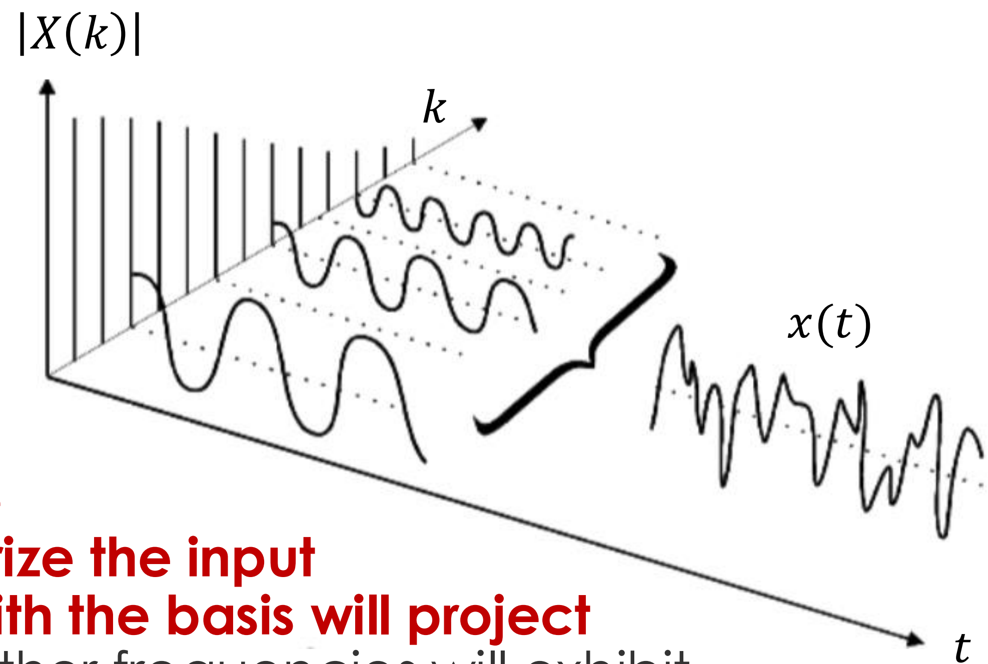
The DFT decomposes the finite-length signal into a set of N sinusoidal components that show “how much” oscillatory behavior at frequency $2\pi k/N$ is contained in the signal.
In this sense, the DFT is the **projection of the observed signal onto the sinusoidal basis set** represented by the N DFT coefficients that spans the observation interval $[-Fs/2, Fs/2]$.

The DFT measurement chain

Discrete Fourier Transform



$$X(k) \triangleq \frac{2}{B} \sum_{n=0}^{N-1} w(n) x(n) W_N^{kn}$$



From the continuum of possible frequencies that can characterize the input signal, only those coinciding with the basis will project onto a single basis vector. All other frequencies will exhibit **non-zero projections** on the entire basis set. This phenomenon is called **spectral leakage** and is the result of processing finite duration records.

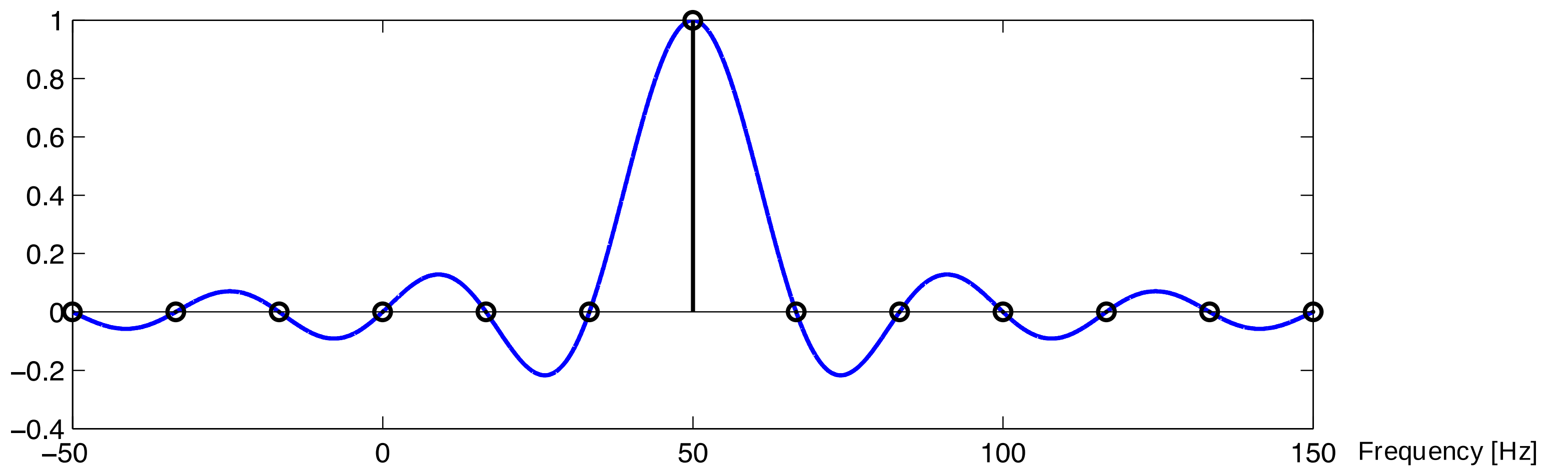
The Spectral Leakage

An analytical interpretation

OBSERVATION: as known, the DFT can be seen as a frequency-discretization of the continuous spectrum. The DFT bins are equally spaced by the quantity $\Delta f = 1/T$ also called **DFT frequency resolution**.

If f_0 is a multiple of the frequency resolution Δf (i.e., **if the window contains an integer number of periods of the signal**), the zero crossing of the translated sinc functions, happens exactly at multiples of $1/T$.

The only frequency that will have a non-zero projection into the DFT basis set, will be the $f = f_0$ (i.e., the DFT bin with index $k = \frac{f_0}{\Delta f}$, see also slide #6).



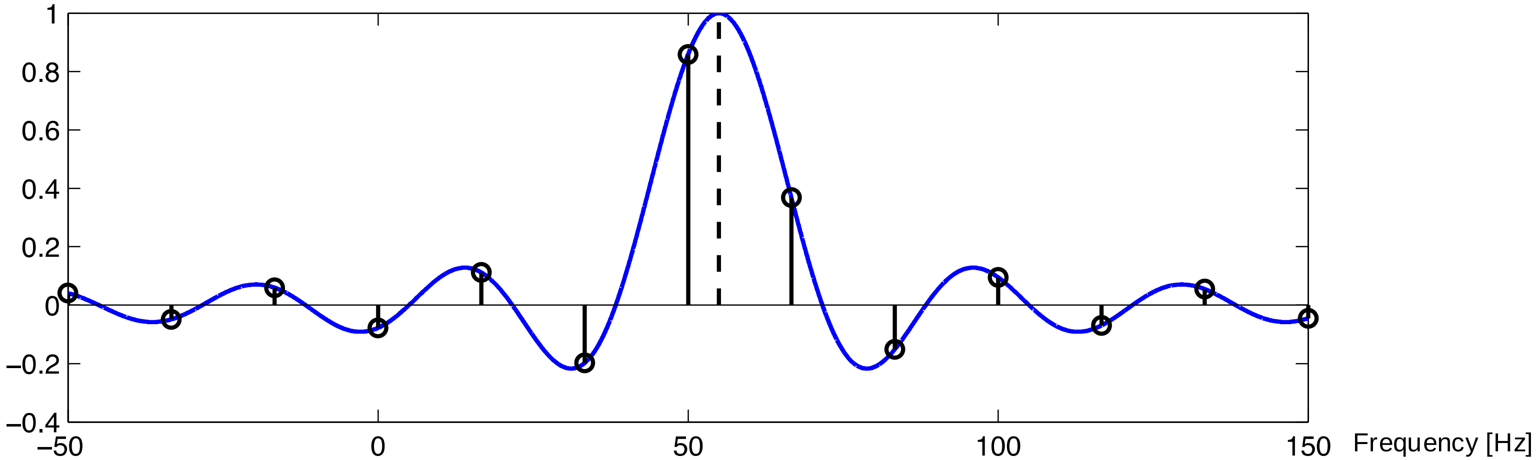
The Spectral Leakage

An analytical interpretation

OBSERVATION: As known, the DFT can be seen as a frequency-discretization of the continuous spectrum. The DFT bins are equally spaced by the quantity $\Delta f = 1/T$ also called **DFT frequency resolution**.

If f_0 is NOT a multiple of the frequency resolution Δf (i.e., **if the window DOES NOT contain an integer number of periods of the signal**), the zero crossing of the translated sinc functions, DOES NOT happen exactly at multiples of $1/T$.

All the frequencies will exhibit non-zero projection on the entire basis set even though the majority of the spectrum energy will be concentrated around $f = f_0$.



The Spectral Leakage

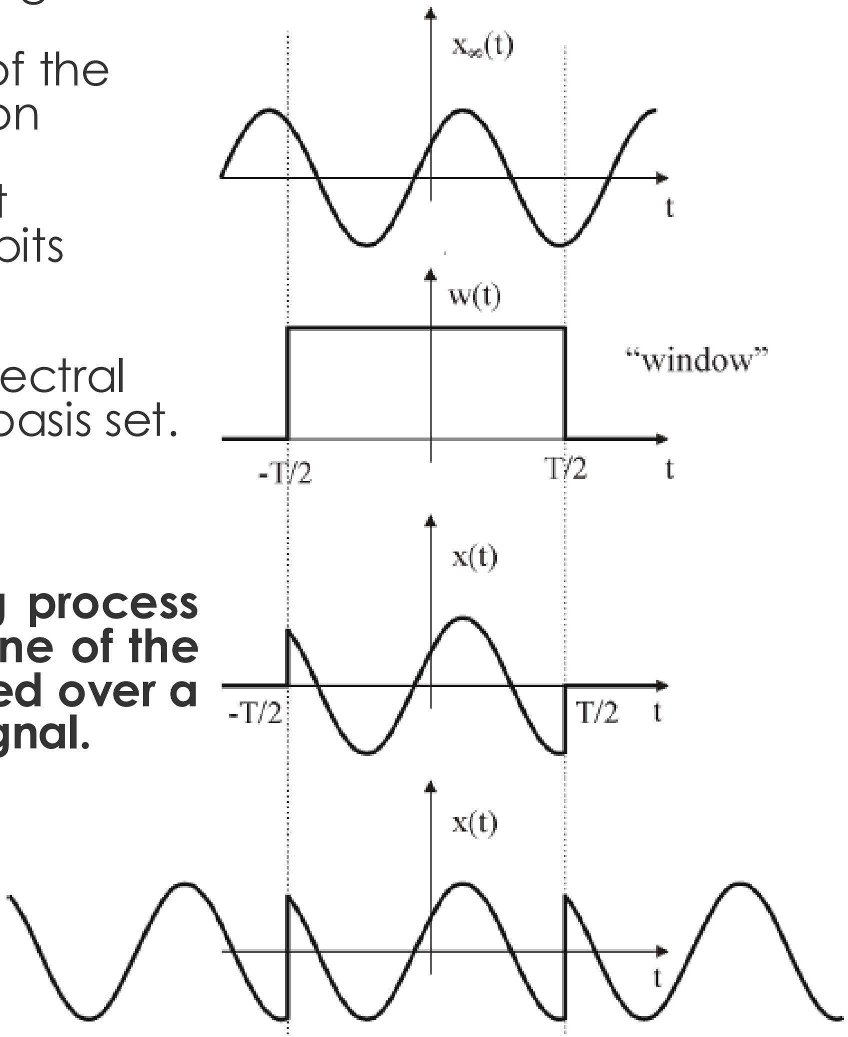
A more intuitive interpretation

An intuitive way to see leakage is the following:

- Signals with frequencies other than those of the basis set, are not periodic in the observation window.
- The periodic extension of a signal does not commensurate with its natural period exhibits discontinuities at the boundaries of the observation.
- The discontinuities are responsible of for spectral contributions (or leakage) over the entire basis set.

THEREFORE

Spectral leakage arises when the sampling process is not synchronized with the fundamental tone of the signal under analysis and the DFT is computed over a non-integer number of cycles of the input signal.



Scalloping Losses

As we have seen, the use of a finite time window implies the frequency bins to be discrete and equally spaced by the quantity $\Delta f = 1/T$. As a result, the tone we may search (e.g. the fundamental frequency component of voltages and currents) might fall within two adjacent bins. This leads to a biased estimate of the corresponding phasor. This phenomenon is known as **scalloping losses**.

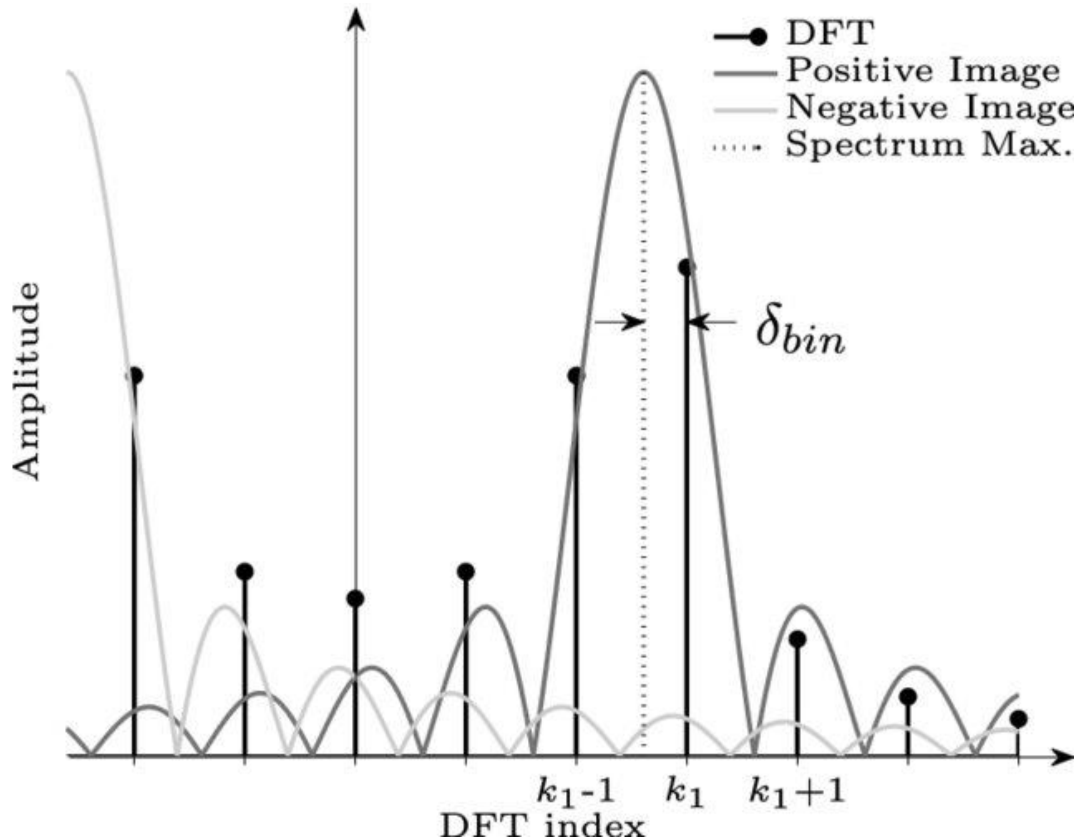
We will see how to fix it by using the interpolation of the DFT bins.

The Spectral Leakage

The long range spectral leakage

It refers to the effects of leakage caused by the tails (or minor lobes) of the **negative spectrum image that interfere on the positive one**.

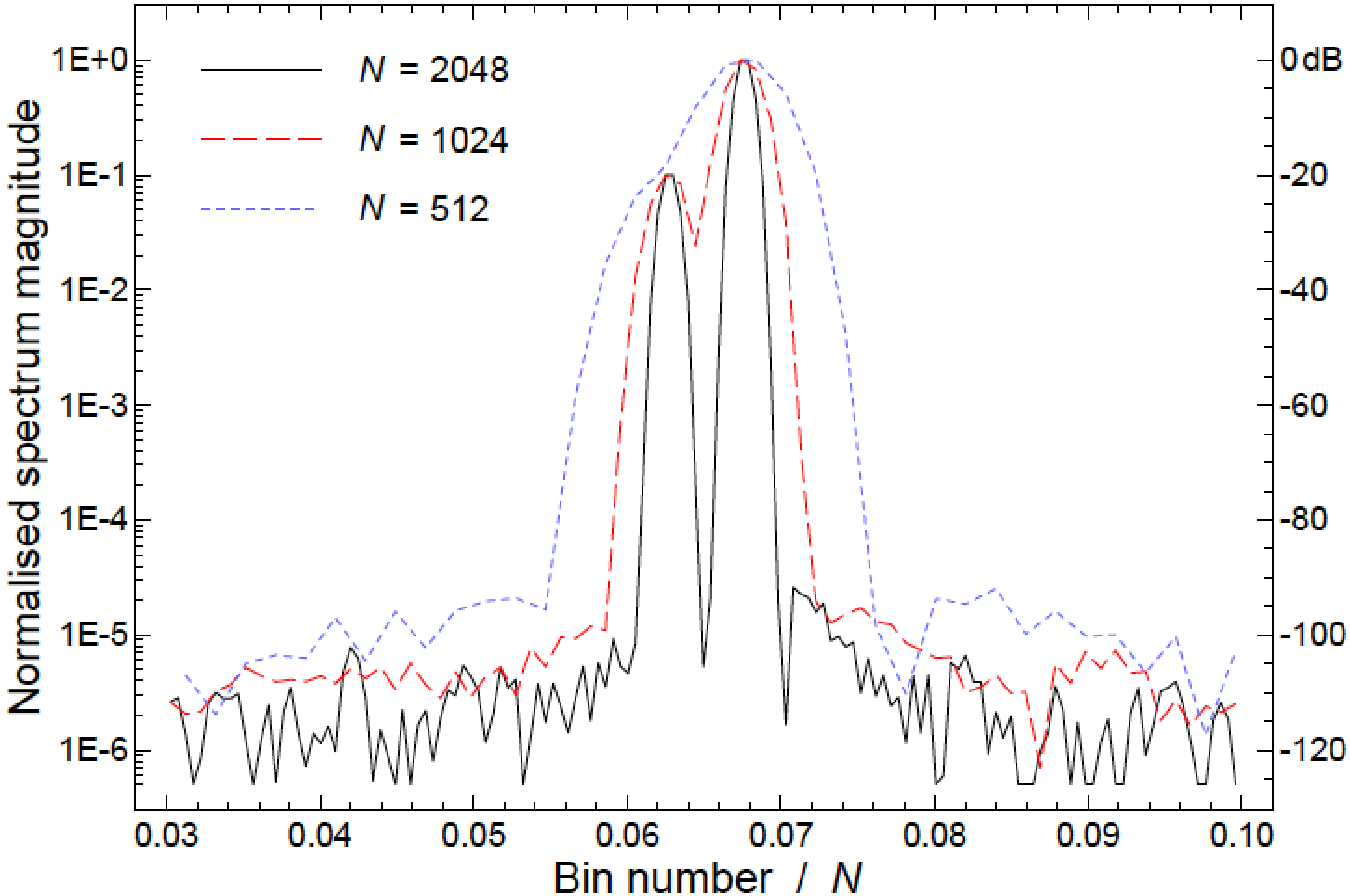
This problem may be very relevant in case we are interested to estimate phasors **whose frequency is close to DC** (as the case of power grid voltage and current signals at 50/60 Hz).



Windowing and Spectral Leakage

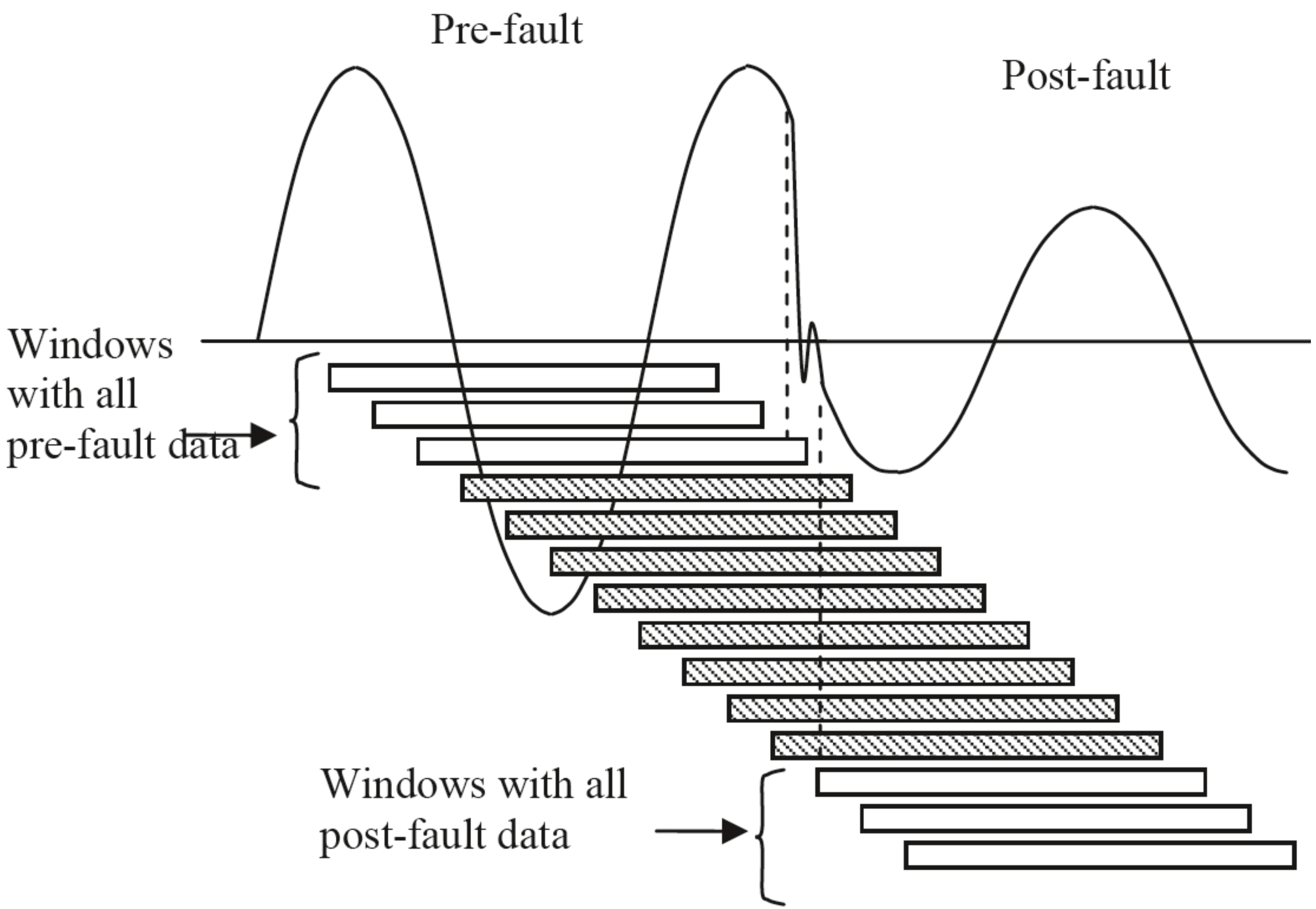
Choosing an appropriate window length.

Example for a signal containing two different tones.



Windowing and Spectral Leakage

Choosing an appropriate window length: the problem of the time-response



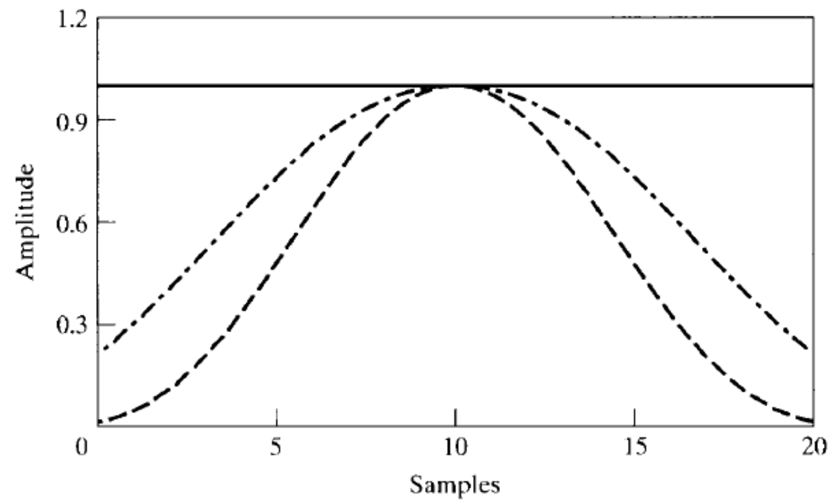
Windowing and Spectral Leakage

Special windowing functions

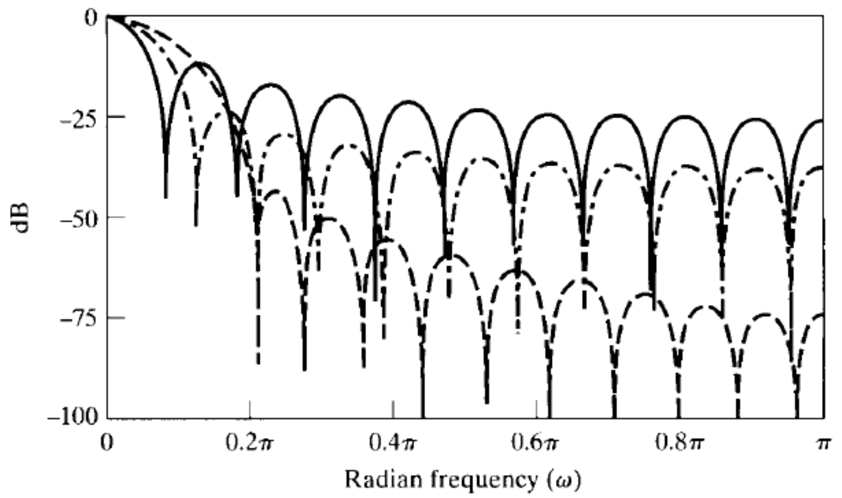
PROBLEM:
In the case of the rectangular window, the spectral leakage is mainly due to the high side-lobes of its Fourier transform caused by the strong discontinuities at the rectangular window edges.

IDEA:
A **triangular window**, reduces the spectral leakage as it brings to zero the 1st order derivative but still has discontinuities at the window center and at the edges for higher order of derivatives.

BETTER IDEA:
A **bell-shaped window** might be even better as it smooth the discontinuities both at the edges and at the center.



(a)

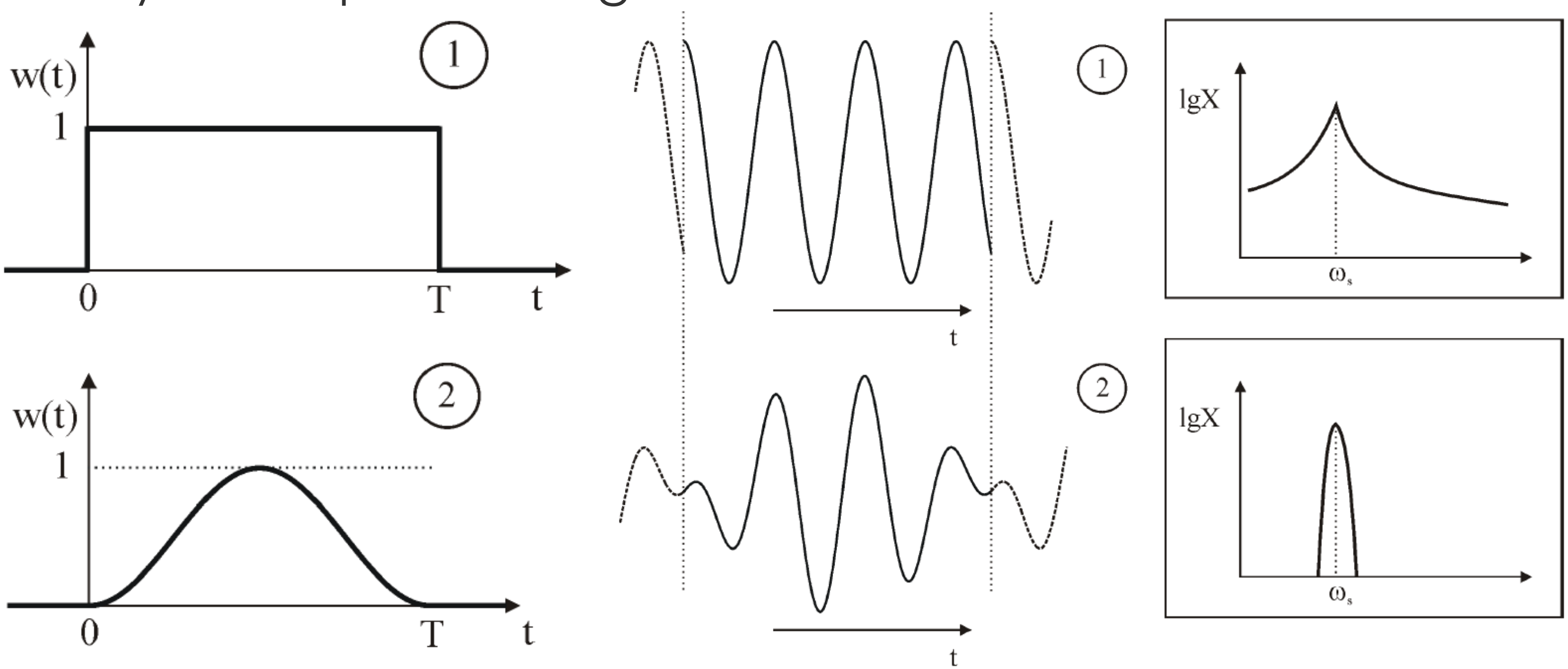


Windowing and Spectral Leakage

Special windowing functions

Windows are weighting function applied to the acquired set of samples to reduce the spectral leakage associated with finite observation intervals.

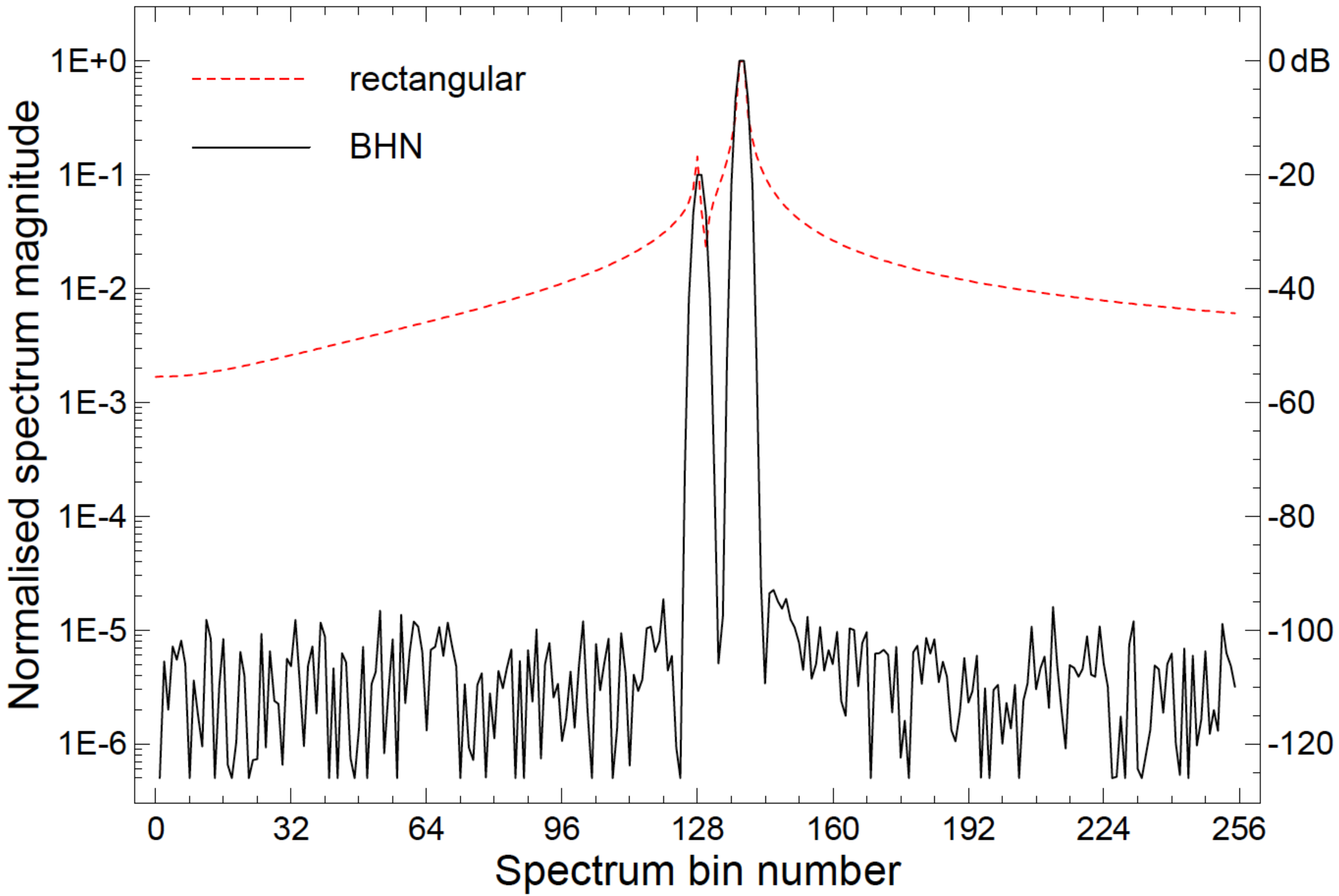
From the viewpoint explained in previous slide, windowing functions are applied to data to reduce the order of discontinuity at the boundary of the periodic signal extension.



Windowing and Spectral Leakage

Special windowing functions

Example for a signal containing two different tones.

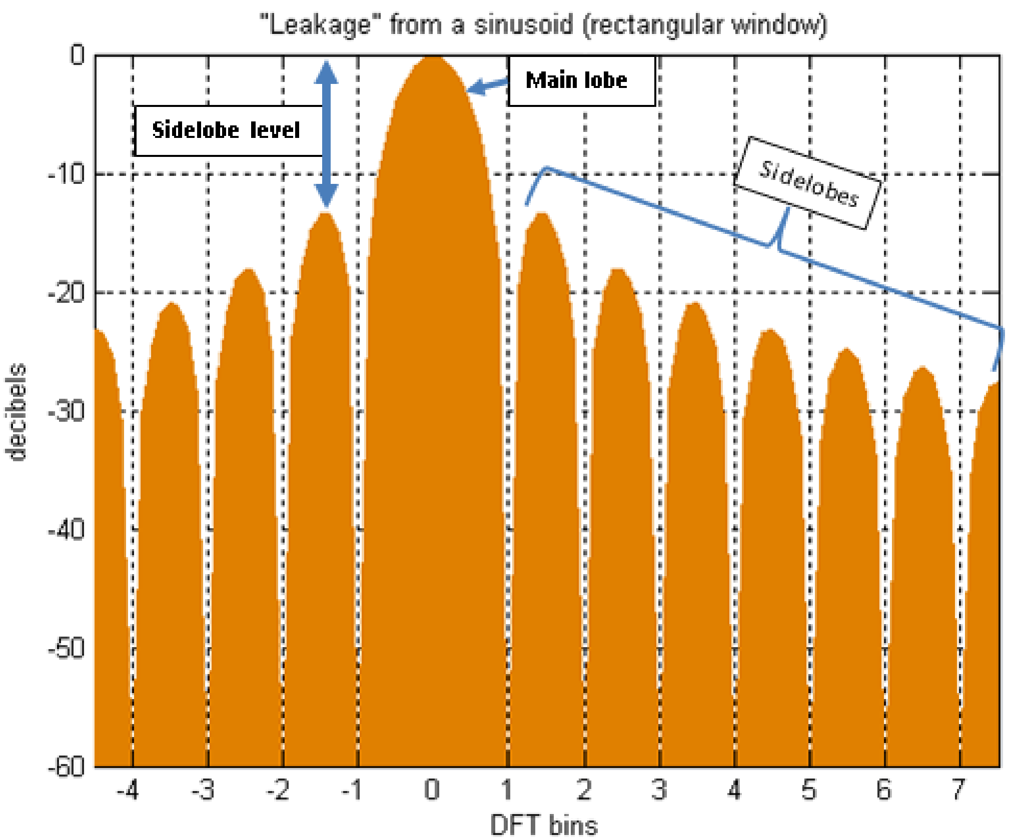


Choosing an appropriate windowing function

Several windowing functions have been defined in the literature trying to minimize the highest number of derivatives at the windows edges. They are well summarized in [1].

The choice of a specific windowing function has to be based on the specific requirements of the targeted application.
The characteristics of each windowing function are usually expressed in terms of:

- **Main lobe 3dB bandwidth**
- **Highest side lobe level**
- **Side-lobe falling rate**
- **Spectral resolution**
- **Scalloping loss**
- Etc.



References

1. **Chapter "DFT-based synchrophasor estimation processes for Phasor Measurement Units applications: algorithms definition and performance analysis", in the book "Advanced Techniques for Power System Modelling, Control and Stability Analysis" edited by F. Milano, IET 2015.**

2. **F. J. Harris, "On the use of windows for harmonic analysis with the Discrete Fourier Transform," Proceedings of the IEEE, vol. 66, no. 1, pp. 51–83, 1978.**

3. G. Heinzel, A. Rudiger and R. Schilling, "Spectrum and spectral density estimation by the Discrete Fourier transform (DFT), including a comprehensive list of window functions and some new flat-top windows".

4. H. A. Gaberson, "A Comprehensive Windows Tutorial"